How to tinker with a homeomorphism and get away with it

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Mary Rees and BCL

Mary Rees published a paper “A minimal positive entropy homeomorphism of the 2-torus” in 1980.

In that paper she gave a construction that allowed the modification of a minimal homeomorphism to suit her purposes. That construction was intricate and hard to understand, so in 2011, F. Béguin, S. Crovisier, and F. Le Roux wrote a 66-page paper, “Construction of curious minimal uniquely ergodic homeomorphisms on manifolds”, one of whose goals was to make the Rees construction more accessible.

Suppose we have a homeomorphism $H : \mathbb{T}^2 \to \mathbb{T}^2$.

By a *rectangle* we mean any subset of $\mathbb{T}^2$ homeomorphic to the unit disc in $\mathbb{R}^2$.

Let $\mathcal{E}, \mathcal{F}$ be a collection of rectangles. We say that $\mathcal{F}$ refines $\mathcal{E}$ if: (a) every element of $\mathcal{E}$ contains at least one element of $\mathcal{F}$; (b) for elements $X \in \mathcal{E}$, $Y \in \mathcal{F}$ either $X \cap Y = \emptyset$ or $Y \subset \operatorname{int} X$. We define

$$\operatorname{mesh} \mathcal{E} = \max\{\operatorname{diam} X : X \in \mathcal{E}\}.$$

p-times iterable

Let $p \in \mathbb{N}$. A collection of rectangles $\mathcal{E}$ is *p-times iterable* if for rectangles $X, Y \in \mathcal{E}$ and integers $-p \leq k, s \leq p$, ether $H^k(X) = H^s(Y)$ or $H^k(X) \cap H^s(Y) = \emptyset$. For any p-times iterable family of rectangles $\mathcal{E}$ and any $0 \leq n \leq p$,

$$\mathcal{E}^n = \bigcup_{|k| \leq n} H^k(\mathcal{E}),$$

where as usual $H(\mathcal{E}) = \{H(X) : X \in \mathcal{E}\}$. In particular, $\mathcal{E}^0 = \mathcal{E}$. Given an integer $0 \leq n \leq p$ we define an oriented graph $G = G(\mathcal{E}^n)$, where the vertices are elements of $\mathcal{E}^n$ and there is an edge from X to Y provided that $H(X) = Y$. For $n < p$ we say that $\mathcal{E}^n$ *has no cycle* if the graph $G(\mathcal{E}^n)$ has no cycle.

$\mathcal{F}$ *is compatible with* $\mathcal{E}$ *for p iterates*

For a collection of rectangles $\mathcal{E}$, let us denote by $\mathfrak{s}(\mathcal{E})$ the union of all rectangles in $\mathcal{E}$. Fix an integer $p \geq 0$ and let $\mathcal{E}, \mathcal{F}$ be collections of rectangles such that $\mathcal{E}$ is p-times iterable and $\mathcal{F}$ is $(p+1)$-times iterable. Assume additionally that $\mathcal{F}^{p+1}$ refines $\mathcal{E}^{p}$. If for every k such that $|k| \leq 2p+1$, we have $H^k(\mathfrak{s}(\mathcal{F})) \cap \mathfrak{s}(\mathcal{E}) \subset \mathfrak{s}(F)$, then we say that $\mathcal{F}$ *is compatible with* $\mathcal{E}$ *for p iterates*.

main axioms from Béguin, Crovisier, Le Roux

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Now we are ready to state the main axioms which are the building blocks in the construction. Let $(\mathcal{E}_n)_{n\in\mathbb{N}_0}$ be a sequence of collections of rectangles. We introduce the following hypotheses:

A_1 : For every $n \in \mathbb{N}_0$

- **a_n** : the collection $\mathcal{E}_n$ is $(n+1)$-times iterable and $\mathcal{E}_n^n$ has no cycle;
- **b_n** : the collection $\mathcal{E}_n^{n+1}$ refines the collection $\mathcal{E}_m^{m+1}$ for every $0 \leq m < n$;
- **c_n** : the collection $\mathcal{E}_{n+1}$ is compatible with $\mathcal{E}_n$ for $n+1$ iterates.

A_3 : $\lim_{n\to\infty} \text{mesh}\, \mathcal{E}_n^n = 0$.

the homeomorphisms M_i

Assume that $(M_n)_{n\in\mathbb{N}}$ is a sequence of homeomorphisms $M_n\colon \mathbb{T}^2 \to \mathbb{T}^2$ and that for every n the homeomorphisms Ψ_n, g_n are defined by:

$$\begin{aligned}\Psi_n &= M_n \circ \ldots \circ M_2 \circ M_1,\\ g_n &= \Psi_n^{-1} \circ H \circ \Psi_n.\end{aligned}$$

Finally we set $\Psi_0 = \mathrm{id}$, $g_0 = H$.

More conditions on the M_n

In addition, assume that the homeomorphisms M_n satisfy the conditions specified below:

B_1 : For every $n \in \mathbb{N}_0$:

$B_{1,n}$: The support of the homeomorphism M_n is contained in the set $\mathcal{E}^{n-1}_{n-1}$, where as usual the support of the homeomorphism M_n is defined by $\operatorname{supp} M_n = \overline{\{x : M_n(x) \neq x\}}$.

B_2 : For every $n \in \mathbb{N}_0$:

$B_{2,n}$: The homeomorphisms M_n and H commute along edges of the graph $G(\mathcal{E}^{n-1}_{n-1})$.

B_3 : Denote $\mathcal{A}_n = \mathcal{E}^{n+1}_n \setminus \mathcal{E}^{n-1}_n$ for every $n \in \mathbb{N}_0$.

$B_{3,n}$: The mesh$\{\Psi^{-1}_{n-1}(X) : X \in \mathcal{A}_n\} < 1/n$;

and, in particular,

$$\lim_{n\to\infty} \operatorname{mesh}\{\Psi^{-1}_{n-1}(X) : X \in \mathcal{A}_n\} = 0.$$

A lemma

The following fact is [BCL, Proposition 3.1]. It ensures proper convergence of the constructed functions.

Lemma

Assume that hypotheses $\mathbf{A}_{1,3}$, $\mathbf{B}_{1,2,3}$ *are satisfied. Then:*

1. *The sequence of homeomorphisms* $(\Psi_n)_{n\in\mathbb{N}}$ *converges uniformly to a continuous surjective map* $\Psi\colon \mathbb{T}^2 \to \mathbb{T}^2$.
2. *The sequence of homeomorphisms* $(g_n)_{n\in\mathbb{N}}$ *converges uniformly to a homeomorphism map* $g\colon \mathbb{T}^2 \to \mathbb{T}^2$ *and* $(g_n^{-1})_{n\in\mathbb{N}}$ *converge uniformly to its inverse* g^{-1}.
3. *The homeomorphism* g *is an extension of* H *by* Ψ, *that is,* $H \circ \Psi = \Psi \circ g$.

another lemma

The following is Proposition 3.4 in BCL.

Lemma

Let $K = \bigcap_{n\in\mathbb{N}} \mathfrak{s}(\mathcal{E}_n)$ and assume that hypotheses $\mathbf{A}_{1,3}$, $\mathbf{B}_{1,2,3}$ are satisfied.

1. *Fix $x \in \mathbb{T}^2$ and suppose that there is $m \in \mathbb{Z}$ such that $x \in H^m(K)$. Let $(X_n)_{n\geq m}$ be the decreasing sequence of rectangles in $\mathcal{E}_n^m$ containing x. Then*

$$\Psi^{-1}(x) = \bigcap_{n\geq m} \Psi_n^{-1}(X_n).$$

2. *For every x which does not belong to the orbit of K the set $\Psi^{-1}(x)$ is a single point.*

minimal noninvertible pseudocircle map

We wish to construct a minimal map on the pseudocircle which is not a homeomorphism. The main step of the construction is the following theorem. We denote the annulus by $\mathbb{A}$.

Theorem

There exists a homeomorphism $g\colon \mathbb{A} \to \mathbb{A}$ *with an invariant pseudocircle* $P \subset \mathbb{A}$ *such that* (g, P) *is minimal and there exists a pseudoarc* $A \subset P$ *such that* $\lim_{|n|\to\infty} \operatorname{diam} g^n(A) = 0$.

main ideas of the proof of the theorem

- Let $H\colon \mathbb{A} \to \mathbb{A}$ be the annulus homeomorphism defined by Handel. In particular, (1) H is a rotation on (both) circles that form the boundary of $\mathbb{A}$, (2) there exists an essential pseudocircle $P \subset \mathbb{A}$ that is a minimal, invariant subset under H, and (3) every point from the interior of $\mathbb{A}$ is attracted by P.
- We can "glue" the boundary of $\mathbb{A}$ to a single circle, call it S, which turns $\mathbb{A}$ into $\mathbb{T}^2$, and now (our modified) $H\colon \mathbb{T}^2 \to \mathbb{T}^2$. If we perturb H to a homeomorphism H' in such a way that H and H' coincide in a neighborhood of S, then we can again "cut back" $\mathbb{T}^2$ to $\mathbb{A}$ obtaining a well defined homeomorphism $H'\colon \mathbb{A} \to \mathbb{A}$.
- In particular, if, in a sufficiently small neighborhood of P there is an H'-invariant set P', which is a hereditarily indecomposable circlelike continuum, then it must also be a pseudocircle.

Step 1. Definition of the maps M_n. To start the construction, fix a pseudoarc $A \subset P$ and a point $p \in A$. There exists a sequence of rectangles $(U_n)_{n\in\mathbb{N}}$, $U_{n+1} \subset \operatorname{int} U_n \subset \mathbb{T}^2$ such that $\bigcap_{n\in\mathbb{N}} U_n = A$. There also exists a decreasing sequence of rectangles $(V_n)_{n\in\mathbb{N}}$ such that (1) $V_n \subset U_n$ for each n, (2) $p \in \operatorname{int} V_n$ for every n, and (3) $\cap V_n = \{p\}$. The pseudocircle P is an invariant set of H without fixed points, the pseudoarcs $H^i(A)$ belong to different composants of P for different $i \in \mathbb{Z}$. In particular $H^i(A) \cap H^j(A) = \emptyset$ for $i \neq j$, hence we may assume that for $|i| \leq 3n$ the sets $H^i(U_n)$ are pairwise disjoint. Furthermore, we may assume that $\operatorname{diam} H^i(V_n) < \frac{1}{n+1}$ for $|i| \leq 3n$. Since the pseudoarc A can be chosen to be arbitrarily small, we may assume that $V_0 = U_0$. Let $\mathcal{E}_0 = \{V_0\}$. Take any $k_1 > 2$ and let $M_1 \colon \mathbb{T}^2 \to \mathbb{T}^2$ be a homeomorphism such that $M_1|_{U_1}$ is a homeomorphism between U_1 and V_{k_1} and $M_1|_{\mathbb{T}^2 \setminus \operatorname{int} U_0} = \operatorname{id}$. Require additionally that $M_1(p) = p$.

Main Theorem

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Theorem

There exists a continuous surjection $G\colon \mathbb{A} \to \mathbb{A}$ with an invariant pseudocircle $P \subset \mathbb{A}$ such that (G, P) is minimal but is not one-to-one.

Thanks so much for listening!